Definition 0.1. The covariant **Yoneda embedding** is the functor

$$\begin{array}{ccc} \mathcal{C} & \qquad & c & \xrightarrow{\;f\;} & d \\ \Big\downarrow{\scriptstyle Y} & & {\scriptstyle Y}\Big\downarrow & & \Big\downarrow{\scriptstyle Y} \\ [\mathcal{C}^{op}, \mathcal{S}et] & & \mathcal{C}(-,c) & \xrightarrow[f\circ -]{} & \mathcal{C}(-,d) \end{array}$$

that maps

- any object $c \in \mathcal{C}$ to the hom-functor $\mathcal{C}(-,c)$
- any morphism $f : c \to d$ to the natural transformation $\alpha : \mathcal{C}(-,c) \Rightarrow \mathcal{C}(-,d)$ with components

$$\alpha_x : \mathcal{C}(x,c) \to \mathcal{C}(x,d)$$
$$u \mapsto f \circ u$$

The contravariant **Yoneda embedding** is the functor

$$\begin{array}{ccc} \mathcal{C}^{op} & \qquad & c & \xrightarrow{\;f\;} & d \\ \Big\downarrow{\scriptstyle Y} & & {\scriptstyle Y}\Big\downarrow & & \Big\downarrow{\scriptstyle Y} \\ [\mathcal{C}, \mathcal{S}et] & & \mathcal{C}(c,-) & \xleftarrow[-\circ f]{} & \mathcal{C}(d,-) \end{array}$$

that maps

- any object $c \in \mathcal{C}$ to the hom-functor $\mathcal{C}(c,-)$
- any morphism $f : c \to d$ to the natural transformation $\alpha : \mathcal{C}(d,-) \Rightarrow \mathcal{C}(c,-)$ with components

$$\alpha_x : \mathcal{C}(d,x) \to \mathcal{C}(c,y)$$
$$u \mapsto u \circ f$$

Corollary 0.1. *The Yoneda embeddings are full and faithful.*

Proof. Consider the contravariant Yoneda embedding[1]. Yoneda lemma in the special case $F = \mathcal{C}(d, -)$ is

$$[\mathcal{C}, \mathcal{S}et](\mathcal{C}(c, -), \mathcal{C}(d, -)) \cong \mathcal{C}(d, c)$$

(for any $c, d \in \mathcal{C}$) it means that the hom-sets in the two categories $\mathcal{C}^{op}$ and $[\mathcal{C}, \mathcal{S}et]$ are isomorphic thus the functor is full and faithful. □

Corollary 0.2. *For any $a, b \in \mathcal{C}$ we have that $a \cong b$ iff $\mathcal{C}(a, -) \cong \mathcal{C}(b, -)$ ($\mathcal{C}(-, a) \cong \mathcal{C}(-, b)$).*
In particular if $a, b \in \mathcal{C}$ represent the same functor then $a \cong b$.

Proof. Using Yoneda lemma we have

$$[\mathcal{C}, \mathcal{S}et](\mathcal{C}(a, -), \mathcal{C}(b, -)) \cong \mathcal{C}(b, a)$$
$$[\mathcal{C}, \mathcal{S}et](\mathcal{C}(b, -), \mathcal{C}(a, -)) \cong \mathcal{C}(a, b)$$

thus

$$\begin{aligned} a \cong b &\iff \mathcal{C}(a, b) \cong \mathcal{C}(b, a) \\ &\iff [\mathcal{C}, \mathcal{S}et](\mathcal{C}(a, -), \mathcal{C}(b, -)) \cong [\mathcal{C}, \mathcal{S}et](\mathcal{C}(b, -), \mathcal{C}(a, -)) \\ &\iff \mathcal{C}(a, -) \cong \mathcal{C}(b, -) \end{aligned}$$

□

The intuition is that each object $a \in \mathcal{C}$ is uniquely specified by the morphisms emanating from it (or impinging into it).

See [1] Ch.2.2, [2] Ch.2.1.3, [3] Ch.18.

See also

- Category Theory II 5.1: Yoneda Embedding - YouTube
- The Yoneda Embedding

[1] E. Riehl, *Category Theory in Context* (Dover, 2015).
[2] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).
[3] B. Milewski, *Category Theory for Programmers* (2019).

[1] the covariant case is analogous