The definition of **limits and colimits** generalizes many universal constructions.

Definition 0.1. Consider a small category $\mathcal{I}$, a **diagram** of **shape** $\mathcal{I}$ in a category $\mathcal{C}$ is a functor

$$D : \mathcal{I} \to \mathcal{C}$$

This is how we pick a pattern in a category: we encode the structure of the pattern in a small category $\mathcal{I}$ and define a functor that matches the pattern in the larger category.

Indexing the objects and morphism of $\mathcal{I}$ the functor is

$$\begin{tikzcd} \mathcal{I} \arrow[d, "D"'] & i \arrow[r, "f_{ij}"] \arrow[d, "D"'] & j \arrow[d, "D"] \\ \mathcal{C} & Di \arrow[r, "Df_{ij}"'] & Dj \end{tikzcd}$$

Matching the pattern into the category leads to

Definition 0.2. A **cone** over a diagram $D : \mathcal{I} \to \mathcal{C}$ with **apex** $c \in \mathcal{C}$ is a natural transformations

$$\alpha : \Delta_c \Rightarrow D$$

A **cocone** under a diagram $D : \mathcal{I} \to \mathcal{C}$ with **nadir** $c \in \mathcal{C}$ is a natural transformation

$$\alpha : D \Rightarrow \Delta_c$$

where $\Delta_c : \mathcal{I} \to \mathcal{C}$ is the constant functor.

The naturality condition for the cone is

$$\begin{tikzcd} \Delta_c i \arrow[r, "\alpha_i"] \arrow[d, "\Delta_c f_{ij}"'] & Di \arrow[d, "Df_{ij}"] \\ \Delta_c j \arrow[r, "\alpha_j"'] & Dj \end{tikzcd}$$

Δ_c is the constant functor so we can collapse the morphism on the left

$$\begin{tikzcd} & Di \arrow[dd, "Df_{ij}"] \\ c \arrow[ur, "\alpha_i"] \arrow[dr, "\alpha_j"'] & \\ & Dj \end{tikzcd}$$

the **legs** of the cone are the components $(\alpha_i : c \to Di)_{i \in \mathcal{I}}$ of the natural transformation.

In order to define the limit we can take the category of cones, find the terminal cone and define that as the limit. This is unsatisfying because we need to define morphisms in the category of cones by imposing commutation relations in an *ad-hoc* way.

Better define the limit by a *natural isomorphism*

1. consider the usual contravariant hom-functor $\mathcal{C}^{op} \to \mathcal{S}et$

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[d, "\mathcal{C}(-{,}\mathbf{Lim}D)"'] \\ \mathcal{S}et \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d] & c' \arrow[d] \\ \mathcal{C}(c, \mathbf{Lim}D) & \mathcal{C}(c', \mathbf{Lim}D) \arrow[l, "- \circ f"] \end{tikzcd}$$

2. consider the contravariant functor $\mathcal{C}^{op} \to \mathcal{S}et$

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[d, "{[\mathcal{I},\mathcal{C}](\Delta_-, D)}"'] \\ \mathcal{S}et \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d] & c' \arrow[d] \\ {[\mathcal{I},\mathcal{C}](\Delta_c, D)} & {[\mathcal{I},\mathcal{C}](\Delta_{c'}, D)} \arrow[l, "- \circ \Delta f"] \end{tikzcd}$$

- it maps any object $c \in \mathcal{C}$ to the hom-set $[\mathcal{I}, \mathcal{C}](\Delta_c, D)$ in the functor category $[\mathcal{I}, \mathcal{C}]$

- it maps any morphism $f : c \to c'$ to a mapping of natural transformations from $\{\beta : \Delta_{c'} \Rightarrow D\}$ to $\{\alpha : \Delta_c \Rightarrow D\}$ by pre-composition

$$\alpha_i = \beta_i \circ f$$

3. considering a natural isomorphism between them we get the definition of limit

Definition 0.3. A functor $D : \mathcal{I} \to \mathcal{C}$ has a **limit Lim**D iff there exists a natural isomorphism

$$\mathcal{C}(-, \mathbf{Lim}D) \cong [\mathcal{I}, \mathcal{C}](\Delta_-, D)$$

The definition of limit is an example of Universal Property.

The naturality square is

$$\begin{tikzcd}
\mathcal{C}(c', \mathbf{Lim}D) \arrow[r, leftrightarrow, "\alpha_{c'}"] \arrow[d] & {[\mathcal{I}, \mathcal{C}](\Delta_{c'}, D)} \arrow[d] \\
\mathcal{C}(c, \mathbf{Lim}D) \arrow[r, leftrightarrow, "\alpha_c"'] & {[\mathcal{I}, \mathcal{C}](\Delta_c, D)}
\end{tikzcd}$$

and since we are in $\mathcal{S}et$ we can check naturality element by element

$$\begin{tikzcd}
u \arrow[r, leftrightarrow, "\alpha_{c'}"] \arrow[d] & \mu_i \arrow[d] \\
v = u \circ f \arrow[r, leftrightarrow, "\alpha_c"'] & \nu_i = \mu_i \circ f
\end{tikzcd}$$

where

- $u : c' \to \mathbf{Lim}D$
- $v : c \to \mathbf{Lim}D$
- $\mu_i : c' \to Di$ is the i-th component of $\mu : \Delta_{c'} \Rightarrow D$
- $\nu_i : c \to Di$ is the i-th component of $\nu : \Delta_c \Rightarrow D$

and we see that the naturality condition amounts to the commutativity of the triangles

$$
\begin{tikzcd}
c \arrow[rr, "f"] \arrow[dr, "\nu_i"'] & & c' \arrow[dl, "\mu_i"] \\
& Di &
\end{tikzcd}
$$

which was required to make the category of cones a category.

The idea is that natural transformations provide a higher level language. They implicitly introduce families of commuting diagrams without the need to specify them individually.

See [1] Ch.3 and also

- Category Theory II 1.2: Limits - YouTube
- Category Theory II 2.2: Limits, Naturality - YouTube
- Limits & Colimits (examples)

[1] E. Riehl, *Category Theory in Context* (Dover, 2015).