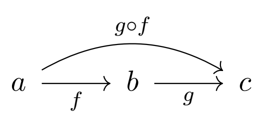
> Atiyah described mathematics as the "science of analogy." In this vein, the purview of category theory is *mathematical analogy*. Category theory provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another. The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline. [1]

Definition 0.1. A **category** is made of the following **data**

- a collection of **objects**: $a, b, c, \ldots$
- a collection of **morphisms**: $f, g, h, \ldots$

[1] such that

- each morphism has a **domain** and a **codomain**: $f : a \to b$
- each object has a designated **identity morphism**: $\mathbf{1}_a : a \to a$
- for any pair of morpisms f, g such that $cod f = dom g$ there exists a designated **composite morphism** $g \circ f$



this **data** obeys the following axioms

- for any $f : a \to b$

$$f \circ \mathbf{1}_a = f = \mathbf{1}_b \circ f$$

- for any composable morphisms f, g, h

$$(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$$

[1]here collection is used to avoid Russel's paradox

> …the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a **functor** and of a **natural transformation**… [2]

See: [2], [1] Ch.1.1, [3] Ch.1.1, [4]. Here are some examples.

Definition 0.2. A category is **small** if the collection of its morphisms is a set (it follows that its objects form a set too).

Definition 0.3. A category is **locally small** if the collection of morphisms between any pair of fixed objects is a set.

Definition 0.4. In a locally small category $\mathcal{C}$, for any pair of objects $a, b \in \mathcal{C}$ the **hom-set** $\mathcal{C}(a, b)$ is the set of morphisms $a \to b$.

Definition 0.5. A morphism $f : a \to b$ is an **isomorphism** iff there exists $g : b \to a$ such that $f \circ g = \mathbf{1}_b$ and $g \circ f = \mathbf{1}_a$. Then the objects a, b are **isomorphic** $a \cong b$.

In a category there is no notion of equality of objects (they can be isomorphic at most), there is equality of morphisms though (which is used extensively in commuting diagrams).

For a discussion on equality in mathematics see [5].

It is easy to see that if $a \cong b$ then if there exists a morphism $c \to a$ (or $a \to c$) then there must exists $c \to b$ (or $b \to c$). Thus for two objects being isomorphic means having the same relationship with the rest of the category.

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[1] E. Riehl, *Category Theory in Context* (Dover, 2015).
[2] S. M. Lane, *Categories for the Working Mathematician* (Springer New York, 1971).
[3] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).
[4] B. Milewski, *Category Theory for Programmers* (2019).
[5] B. mazur, *When Is One Thing Equal to Some Other Thing?*, (2006).