The intuition is that a monad is a consistent choice of formal expressions together with ways to evaluate them.

A formal expression is an arrangement of formal variables that may not have a result but obeys manipulation rules. The construction of formal expression is *functorial* because we can always apply a function to the formal variables[1].

The monad structure then comes out naturally because we can always

- form the trivial expression out of a single formal variable (the monad unit)
- simplify a composite expression (the monad multiplication)

Now we want to equip the monad with a way to evaluate formal expressions that is to substitute values into the formal variables and get a value back. For this we need an algebra

Definition 0.1. Consider a monad (T, η, μ) in a category $\mathcal{C}$ a **monad algebra**[2] is an algebra $(c, \epsilon : Tc \to c)$ that is compatible with the monad structure, that is the diagrams

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & Tc \\ & \searrow^{\mathbf{1}_c} & \downarrow \epsilon \\ & & c \end{array} \qquad \begin{array}{ccc} TTc & \xrightarrow{\mu_c} & Tc \\ T\epsilon \downarrow & & \downarrow \epsilon \\ Tc & \xrightarrow[\epsilon]{} & c \end{array}$$

commute.

Note that when dealing with an algebra (which has a specific carrier object) we always work with the *components* of the natural transformations η, μ.

Proposition 0.1. *Monad algebras for a monad (T, η, μ) on a category $\mathcal{C}$ form the* ***Eilenberg-Moore*** *category $\mathcal{C}^T$.*

---

[1] for example formal sums of variables: $x \in X$, $FX = x_1 + \cdots + x_n$ we make F functorial with $Ff(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$

[2] Eilenberg-Moore algebra

Definition 0.2. Consider a monad (T, η, μ) on a category $\mathcal{C}$ and an object $c \in \mathcal{C}$ then (Tc, μ_c)[3] is a **free monad algebra**.

(Tc, μ_c) is a monad algebra because

- the first compatibility diagram

$$\begin{array}{ccc} Tc & \xrightarrow{\eta} & TTc \\ & \searrow^{\mathbf{1}_{Tc}} & \downarrow{\mu} \\ & & Tc \end{array}$$

  is just the left unitality

- the second compatibility diagram

$$\begin{array}{ccc} TTTc & \xrightarrow{\mu_{Tc}} & TTc \\ {\scriptstyle T\mu_c}\downarrow & & \downarrow{\scriptstyle \mu_c} \\ TTc & \xrightarrow[\mu_c]{} & Tc \end{array}$$

  is just the associativity

See [1] Ch.5.2.

[1] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).

[3] note that $\mu_c : T(Tc) \to (Tc)$ is an evaluation map