The intuition is that a monad is a consistent way to extend spaces to include generalized elements and functions.

A generalized element can be the result of some operation that might

- fail (`Maybe`)
- produce multiple outcomes (`List`)
- produce multiple outcomes with different probabilities
- produce some outcome together with side effects

in all these cases we might replace morphisms $f : a \to b$ with $f : a \to Tb$

Definition 0.1. Let (T, η, μ) be a monad on a category $\mathcal{C}$, a **Kleisli morphism** $a \to b$ of T is a morphism $f : a \to Tb$ in $\mathcal{C}$

Taking the naturality square of η

$$\begin{array}{ccc} a & \xrightarrow{\eta_a} & Ta \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ b & \xrightarrow{\eta_b} & Tb \end{array}$$

note that any $f : a \to b$ uniquely defines a map $\eta_b \circ f : a \to Tb$.

This is different from *extending* existing morphisms to $T-$, we are allowing more general morphisms that may give values in Tb that may not come from b alone.

Definition 0.2. Let (T, η, μ) be a monad on a category $\mathcal{C}$, let $f : a \to Tb$ and $g : b \to Tc$, the **Kleisli composition** $g \circ_T f : a \to Tc$ is given by

$$a \xrightarrow{f} Tb \xrightarrow{Tg} TTc \xrightarrow{\mu_c} Tc$$

$$g \circ_T f = \mu_c \circ Tg \circ f$$

Definition 0.3. Let (T, η, μ) be a monad over a category $\mathcal{C}$, the **Kleisli category** $\mathcal{C}_T$ has

- the same objects of $\mathcal{C}$
- a Kleisli morphisms $f : a \to b$ for any $f : a \to b$ in $\mathcal{C}$
- identity morphisms $\mathbf{1}_a = \eta_a : a \to Ta$
- the Kleisli composition

See [1] Ch.5.1.

[1] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).