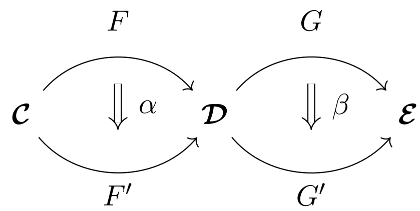
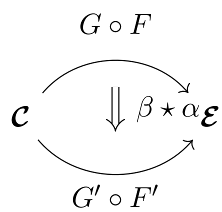
Theorem 0.1. *Given a pair of natural transformations* $\alpha : F \Rightarrow F'$ *and* $\beta : G \Rightarrow G'$



the ***horizontal composition*** $\beta \star \alpha : G \circ F \Rightarrow G' \circ F'$



has components

$$(\beta \star \alpha)_c = \beta_{F'c} \circ G\alpha_c$$

Proof. Consider the two naturality squares ($c \in \mathcal{C}, d \in \mathcal{D}$)

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & F'c \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle F'f} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & F'c' \end{array} \qquad \begin{array}{ccc} Gd & \xrightarrow{\beta_d} & G'd \\ {\scriptstyle Gg}\downarrow & & \downarrow{\scriptstyle G'g} \\ Gd' & \xrightarrow[\beta_{d'}]{} & G'd' \end{array}$$

we proceed in this way

1. lift the first square with G
2. apply the second square on the right setting $d = F'c, d' = F'c', g = F'f$

$$\begin{array}{ccccc} G(Fc) & \xrightarrow{G\alpha_c} & G(F'c) & \xrightarrow{\beta_{F'c}} & G'(F'c) \\ {\scriptstyle G(Ff)}\downarrow & & \downarrow{\scriptstyle G(F'f)} & & \downarrow{\scriptstyle G'(F'f)} \\ G(Fc') & \xrightarrow[G\alpha_{c'}]{} & G(F'c') & \xrightarrow[\beta_{F'c'}]{} & G'(F'c') \end{array}$$

the whole rectangle commutes and we get

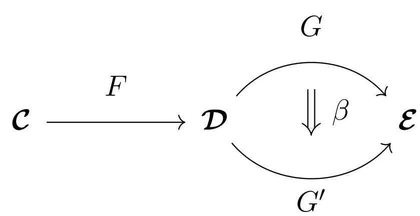
$$\begin{array}{ccc} (G\circ F)c & \xrightarrow{(\beta\star\alpha)_c} & (G'\circ F')c \\ {\scriptstyle (G\circ F)f}\downarrow & & \downarrow{\scriptstyle (G'\circ F')f} \\ (G\circ F)c' & \xrightarrow[(\beta\star\alpha)_{c'}]{} & (G'\circ F')c' \end{array}$$

with

$$(\beta\star\alpha)_c = \beta_{F'c}\circ G\alpha_c$$

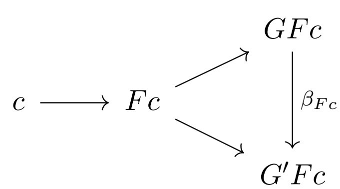
□

Example 0.1. *(**Left Whiskering**) Consider the case* $\alpha = \mathbf{1}_F$

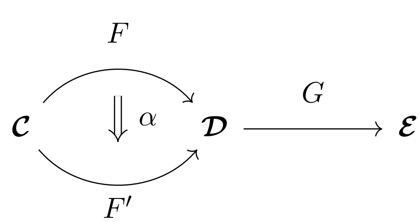


$$(\beta\star\mathbf{1}_F)_c = (\beta\circ F)_c = \beta_{Fc}$$

Alternatively, taking the c *component*

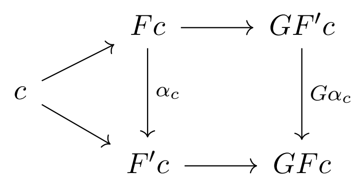


Example 0.2. *(**Right Whiskering**) Consider the case* $\beta = \mathbf{1}_G$



$$(\mathbf{1}_G \star \alpha)_c = (G \circ \alpha)_c = G\alpha_c$$

Alternatively, taking the c component



See [1] Ch.1.7 and also

- Category Theory II 6.1: Examples of Adjunctions - YouTube

[1] E. Riehl, *Category Theory in Context* (Dover, 2015).