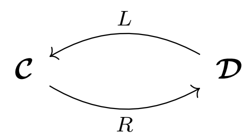
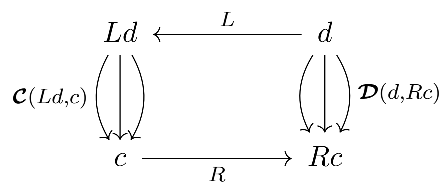
Consider a weaker notion of equivalence of categories, instead of taking full isomorphisms take only *half*





Definition 0.1. (Adjunction I) An **adjunction** $L \dashv R$ consists of a pair of functors $R : \mathcal{C} \to \mathcal{D}, L : \mathcal{D} \to \mathcal{C}$ together with a natural isomorphism[1] between the hom-functors

$$\mathcal{C}(Ld, c) \cong \mathcal{D}(d, Rc)$$

for any $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

The corresponding morphisms

$$Ld \xrightarrow{f^\sharp} c \qquad \leftrightarrow \qquad d \xrightarrow{f^\flat} Rc$$

through the bijection are **adjoint** of each other.

The intuition is that we can get the all information about $c \in \mathcal{C}$ by first mapping it with $R : \mathcal{C} \to \mathcal{D}$ and then analysing it in category $\mathcal{D}$ (from any $d \in \mathcal{D}$), the natural isomorphism guarantees that this information is the same as that extracted directly in $\mathcal{C}$ (from any $Ld \in \mathcal{C}$).

See [1] Ch.4, [2] Ch.4, [3] Ch.18, [4] Ch.IV.
See also

- Category Theory II 5.2: Adjunctions - YouTube
- Adjunctions (examples)

[1]that is an isomorphism between the hom-sets that is also a natural transformation between hom-functors when varying both c and d

1 Universal Property & the Unit

Taking the adjunction $L \dashv R$ and fixing $d \in \boldsymbol{\mathcal{D}}$ we have the natural isomorphism between hom-functors

$$\boldsymbol{\mathcal{C}}(Ld, -) \cong \boldsymbol{\mathcal{D}}(d, R-)$$

thus the hom-functor $\boldsymbol{\mathcal{D}}(d, R-) : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{S}et}$ together with $Ld \in \boldsymbol{\mathcal{C}}$ form a universal property.

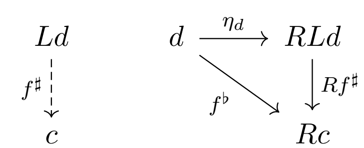
As usual specializing the naturality square ($f^\sharp : Ld \to c$)

$$\begin{array}{ccc} \boldsymbol{\mathcal{C}}(Ld, Ld) & \xleftrightarrow{\alpha_{Ld}} & \boldsymbol{\mathcal{D}}(d, RLd) \\ {\scriptstyle f^\sharp \circ -}\downarrow & & \downarrow{\scriptstyle Rf^\sharp \circ -} \\ \boldsymbol{\mathcal{C}}(Ld, c) & \xleftrightarrow[\alpha_c]{} & \boldsymbol{\mathcal{D}}(d, Rc) \end{array}$$

and evaluating element-wise ($f^\flat = \alpha_c f^\sharp$)

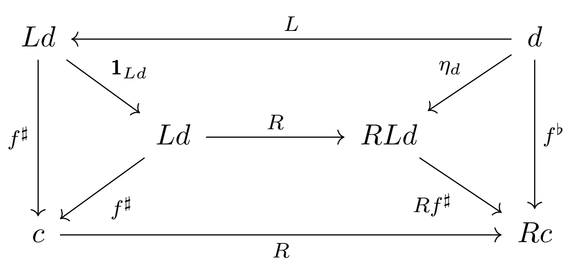
$$\begin{array}{ccc} \mathbf{1}_{Ld} & \xleftrightarrow{\alpha_{Ld}} & \eta_d = \alpha_{Ld}\mathbf{1}_{Ld} \\ \downarrow & & \downarrow{\scriptstyle Rf^\sharp \circ -} \\ f^\sharp & \xleftrightarrow[\alpha_c]{} & f^\flat = Rf^\sharp \circ \eta_d \end{array}$$

we have the existence and uniqueness condition: for any $f^\flat : d \to Rc$ there exists a unique $f^\sharp : Ld \to c$ such that $Rf^\sharp \circ \eta_d = f^\flat$



Note that the map $\eta_d : d \to RLd$ is the adjoint of $\mathbf{1}_{Ld}$.

Looking at the full diagram



we see that the identity diagram for $\mathbf{1}_{Ld}$ on the left induces commuting diagrams on the right, this can be made more precise in the following

Lemma 1.1. *Consider the adjunction $L \dashv R$ and the diagrams*

$$\begin{array}{ccc} Ld & \xrightarrow{Lh} & Ld' \\ {\scriptstyle f^\sharp}\downarrow & & \downarrow{\scriptstyle g^\sharp} \\ c & \xrightarrow[k]{} & c' \end{array} \qquad \begin{array}{ccc} d & \xrightarrow{h} & d' \\ {\scriptstyle f^\flat}\downarrow & & \downarrow{\scriptstyle g^\flat} \\ Rc & \xrightarrow[Rk]{} & Rc' \end{array}$$

the diagram on the left commutes if and only if the diagram on the right commutes.[2]

Proof. The naturality square for fixed $c' \in \mathcal{C}$ ($h : d \to d'$) is

$$\begin{array}{ccc} \mathcal{C}(Ld', c') & \xrightarrow{\flat} & \mathcal{D}(d', Rc') \\ {\scriptstyle -\circ Lh}\downarrow & & \downarrow{\scriptstyle -\circ h} \\ \mathcal{C}(Ld, c') & \xrightarrow[\flat]{} & \mathcal{D}(d, Rc') \end{array}$$

$$(g^\sharp \circ Lh)^\flat = g^\flat \circ h$$

The naturality square for fixed $d \in \mathcal{D}$ ($k : c \to c'$) is

$$\begin{array}{ccc} \mathcal{C}(Ld, c) & \xrightarrow{\flat} & \mathcal{D}(d, Rc) \\ {\scriptstyle k\circ -}\downarrow & & \downarrow{\scriptstyle Rk\circ -} \\ \mathcal{C}(Ld, c') & \xrightarrow[\flat]{} & \mathcal{D}(d, Rc') \end{array}$$

$$(k \circ f^\sharp)^\flat = Rk \circ f^\flat$$

Suppose the left diagram commutes

$$g^\sharp \circ Lh = k \circ f^\sharp$$

substituting into naturality

[2] the converse is also true: equivalence between commuting diagrams on both sides implies the naturelity of the bijections.

$$g^\flat \circ h = Rk \circ f^\flat$$

thus the right diagram also commutes (the converse is similar).
□

Proposition 1.1. *The maps $\eta_d : d \to RLd$ form a natural transformation*

$$\eta : \mathbf{1}_{\mathcal{D}} \Rightarrow R \circ L$$

of endofunctors $\mathcal{D} \to \mathcal{D}$.

Proof. The naturality square for η ($f : d \to d'$) is

$$\begin{array}{ccc} d & \xrightarrow{\eta_d} & RLd \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle RLf} \\ d' & \xrightarrow{\eta_{d'}} & RLd' \end{array}$$

using the previous lemma this diagram commutes iff the corresponding diagram in $\mathcal{C}$

$$\begin{array}{ccc} Ld & \xrightarrow{\mathbf{1}_d} & Ld \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle Lf} \\ Ld' & \xrightarrow[\mathbf{1}_{d'}]{} & Ld' \end{array}$$

commutes, which is clearly the case.
□

Definition 1.1. The natural transformation

$$\eta : \mathbf{1}_{\mathcal{D}} \Rightarrow R \circ L$$

is called the **unit** of the adjunction $L \dashv R$.

2 Universal Property & the Counit

Taking the adjunction $L \dashv R$ and fixing $c \in \boldsymbol{\mathcal{C}}$ we have the natural isomorphism between hom-functors

$$\boldsymbol{\mathcal{C}}(L-, c) \cong \boldsymbol{\mathcal{D}}(-, Rc)$$

thus the hom-functor $\boldsymbol{\mathcal{C}}(L-, c) : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{S}et}$ together with $Rc \in \boldsymbol{\mathcal{D}}$ form a universal property.

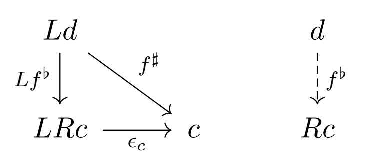
As usual specializing the naturality square ($f^\flat : d \to Rc$)

$$\begin{array}{ccc} \boldsymbol{\mathcal{D}}(Rc, Rc) & \xrightarrow{\alpha_{Rc}} & \boldsymbol{\mathcal{C}}(LRc, c) \\ {\scriptstyle -\circ f^\flat}\downarrow & & \downarrow{\scriptstyle -\circ Lf^\flat} \\ \boldsymbol{\mathcal{D}}(d, Rc) & \xrightarrow[\alpha_d]{} & \boldsymbol{\mathcal{C}}(Ld, c) \end{array}$$

and evaluating element-wise ($f^\sharp = \alpha_c f^\flat$)

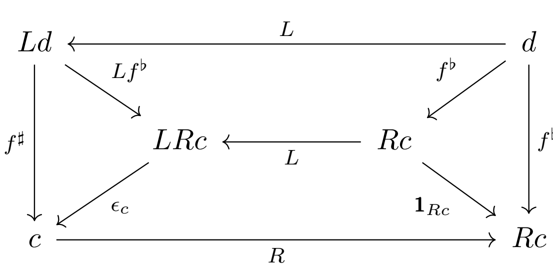
$$\begin{array}{ccc} \mathbf{1}_{Rc} & \xrightarrow{\alpha_{Rc}} & \epsilon_c = \alpha_{Rc}\mathbf{1}_{Rc} \\ \downarrow & & \downarrow \\ f^\flat & \xrightarrow[\alpha_d]{} & f^\sharp = \epsilon_c \circ Lf^\flat \end{array}$$

we have the existence and uniqueness condition: for any $f^\sharp : Ld \to c$ there exists a unique $f^\flat : d \to Rc$ such that $f^\sharp = \epsilon_c \circ Lf^\flat$



Note that the map $\epsilon_c : LRc \to c$ is the adjoint of $\mathbf{1}_{Rc}$.

The full diagram is



Proposition 2.1. *The maps $\epsilon_c : c \to LRc$ form a natural transformation*

$$\epsilon : L \circ R \Rightarrow \mathbf{1}_{\mathcal{C}}$$

of endofunctors $\mathcal{C} \to \mathcal{C}$.

Proof. The naturality square for ϵ $(f : c \to c')$ is

$$\begin{array}{ccc} LRc & \xrightarrow{\epsilon_c} & c \\ {\scriptstyle LRf}\downarrow & & \downarrow{\scriptstyle f} \\ LRc' & \xrightarrow[\epsilon_{c'}]{} & c' \end{array}$$

using the previous lemma this diagram commutes iff the corresponding diagram in $\mathcal{D}$

$$\begin{array}{ccc} Rc & \xrightarrow{\mathbf{1}_{Rc}} & Rc \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Rf} \\ Rc' & \xrightarrow[\mathbf{1}_{Rc'}]{} & Rc' \end{array}$$

commutes, which is clearly the case.
□

Definition 2.1. The natural transformation

$$\epsilon : L \circ R \Rightarrow \mathbf{1}_{\mathcal{C}}$$

is called the **counit** of the adjunction $L \dashv R$.

3 Alternative Definition

An alternative definition of the adjunction is

Definition 3.1. (Adjunction II) An **adjunction** $L \dashv R$ consists of a pair of functors $R : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}, L : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ together with two natural transformations

- $\eta : \mathbf{1}_{\boldsymbol{\mathcal{D}}} \Rightarrow R \circ L$ (the **unit**)
- $\epsilon : L \circ R \Rightarrow \mathbf{1}_{\boldsymbol{\mathcal{C}}}$ (the **counit**)

such that

$$
\begin{array}{ccc}
L & \xrightarrow{L \circ \eta} & L \circ R \circ L \\
& \searrow\!\!\!\!= & \downarrow{\scriptstyle \epsilon \circ L} \\
& & L
\end{array}
\qquad\qquad
\begin{array}{ccc}
R & \xrightarrow{\eta \circ R} & R \circ L \circ R \\
& \searrow\!\!\!\!= & \downarrow{\scriptstyle R \circ \epsilon} \\
& & R
\end{array}
$$

commute in the functor category $[\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}]$.

Note that $R \circ L$ and $L \circ R$ are endo-functors, they are like *containers*.

The naturality square for unit ($f : x \to y$ in $\boldsymbol{\mathcal{D}}$) is[3]

$$
\begin{array}{c}
x \\
{\scriptstyle f}\downarrow \\
y
\end{array}
\qquad
\begin{array}{ccc}
x & \xrightarrow{\eta_x} & (R \circ L)x \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (R \circ L)f} \\
y & \xrightarrow[\eta_y]{} & (R \circ L)y
\end{array}
$$

η_x picks a particular element inside $(R \circ L)x$.

The naturality square for counit ($f : x \to y$ in $\boldsymbol{\mathcal{C}}$) is

$$
\begin{array}{c}
x \\
{\scriptstyle f}\downarrow \\
y
\end{array}
\qquad
\begin{array}{ccc}
(L \circ R)x & \xrightarrow{\epsilon_x} & x \\
{\scriptstyle (L \circ R)f}\downarrow & & \downarrow{\scriptstyle f} \\
(L \circ R)y & \xrightarrow[\epsilon_y]{} & y
\end{array}
$$

ϵ_x extracts a particular element from $(L \circ R)x$.

[3] note that $\mathbf{1}_{\boldsymbol{\mathcal{D}}} x = x, \mathbf{1}_{\boldsymbol{\mathcal{D}}} f = f$

Proposition 3.1. *The two definitions of adjunction are equivalent.*

Proof. Consider the isomorphism in the first definition and pick $c = Ld$ then

$$\boldsymbol{\mathcal{C}}(Ld, Ld) \cong \boldsymbol{\mathcal{D}}(d, (R \circ L)d)$$

the identity $\mathbf{1}_{Ld} \in \boldsymbol{\mathcal{C}}(Ld, Ld)$ picks a mapping (the unit)

$$\eta_d : d \to (R \circ L)d$$

which can be seen to be the component of a natural transformation from the naturality of the isomorphism.

Taking $d = Rc$ we get

$$\boldsymbol{\mathcal{C}}((L \circ R)c, c) \cong \boldsymbol{\mathcal{D}}(Rc, Rc)$$

the identity $\mathbf{1}_{Rc}$ picks a mapping (the counit)

$$\epsilon_c : (L \circ R)c \to c$$

which is natural.

In the other direction take

- $\eta_d : d \to (R \circ L)d$
- $\epsilon_c : (L \circ R)c \to c$

and use R and L functors on morphisms.
□

[1] E. Riehl, *Category Theory in Context* (Dover, 2015).
[2] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).
[3] B. Milewski, *Category Theory for Programmers* (2019).
[4] S. M. Lane, *Categories for the Working Mathematician* (Springer New York, 1971).