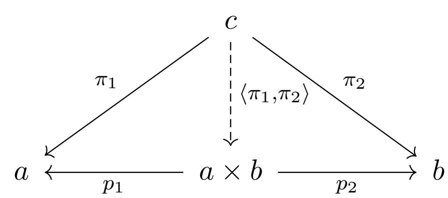
Products and coproducts of two objects in a category are defined by universal construction

Definition 0.1. In a category $\mathcal{C}$ a **product** of two objects $a, b \in \mathcal{C}$ consists in

- an object $a \times b \in \mathcal{C}$
- two morphisms $p_1 : a \times b \to a$ and $p_2 : a \times b \to b$ (the **projections**)

such that for any object c and morphisms $\pi_1 : c \to a, \pi_2 : c \to b$ there exists a unique morphism $\langle \pi_1, \pi_2 \rangle : c \to a \times b$ such that



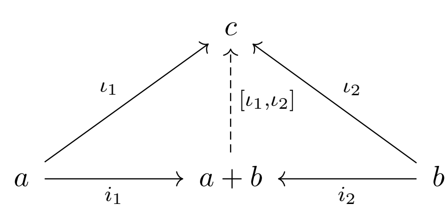
commutes.

The unique morphism $\langle \pi_1, \pi_2 \rangle$ takes all the *ugly and flawed* parts of the candidate product and allows the *nice and clean* projections to do their job.

Definition 0.2. In a category $\mathcal{C}$ a **coproduct** of two objects $a, b \in \mathcal{C}$ consists in

- an object $a + b \in \mathcal{C}$
- two morphisms $i_1 : a \to a + b$ and $i_2 : b \to a + b$ (the **injections**)

such that for any object c and morphisms $\iota_1 : a \to c, \iota_2 : b \to c$ there exists a unique morphism $[\iota_1, \iota_2] : a + b \to c$ such that



commutes.

Definition 0.3. A **cartesian** (**cocartesian**) category has products (coproducts) for every pair of objects.

Example 0.1. *In $\mathcal{S}et$*

- *the product is the cartesian product* $A \times B$
- *the coproduct is the disjoint union* $A \sqcup B$

Example 0.2. *In $\mathcal{H}ask$*

- *the product is* `(a,b)` *the two projections are*

  ```
  fst :: (a,b) -> a
  fst (x,_) = x
  snd :: (a,b) -> b
  snd (_,y) = y
  ```

- *the coproduct is*

  ```
  data Either a b = Left a
                  | Right b
  ```

  `Left` *and* `Right` *corresponds to the two injections*

See [1] Ch.5 and also

- Category Theory 4.2: Products - YouTube
- Category Theory 5.1: Coproducts, sum types - YouTube

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[1] B. Milewski, *Category Theory for Programmers* (2019).
