Theorem 0.1. *(Yoneda lemma) For any functor $F : \mathcal{C} \to \mathcal{S}et$ whose domain $\mathcal{C}$ is locally small and any object $c \in \mathcal{C}$ there is a one-to-one correspondence*

$$[\mathcal{C}, \mathcal{S}et](\mathcal{C}(c,-), F) \cong Fc$$

between natural transformations $\alpha : \mathcal{C}(c,-) \Rightarrow F$ and elements of Fc.[1]

Moreover the correspondence is natural in c and F.

> Yoneda lemma says that hom-functors are special: they have natural transformations to <u>any</u> other functor. And these natural transformations are <u>limited</u> in the sense that they are enumerated by a set.

Proof of correspondence. Consider a natural transformation $\alpha : \mathcal{C}(c,-) \Rightarrow F$

1. the naturality square is

$$\begin{array}{ccccc} x & \quad & \mathcal{C}(c,x) & \xrightarrow{\alpha_x} & Fx \\ f\big\downarrow & & \mathcal{C}(c,f)\big\downarrow & & \big\downarrow Ff \\ y & & \mathcal{C}(c,y) & \xrightarrow[\alpha_y]{} & Fy \end{array}$$

2. specialize it to the case $x = c$

$$\begin{array}{ccccc} c & \quad & \mathcal{C}(c,c) & \xrightarrow{\alpha_c} & Fc \\ f\big\downarrow & & \mathcal{C}(c,f)\big\downarrow & & \big\downarrow Ff \\ y & & \mathcal{C}(c,y) & \xrightarrow[\alpha_y]{} & Fy \end{array}$$

3. the naturality condition is

$$\alpha_y \circ \mathcal{C}(c,f) = Ff \circ \alpha_c$$

[1] there is also the *dual* version $[\mathcal{C}, \mathcal{S}et](\mathcal{C}(-,c), F) \cong Fc$

it holds between <u>sets</u> so it can be *evaluated* point-wise, in particular for $\mathbf{1}_c \in \mathcal{C}(c, c)$

$$[\alpha_y \circ \mathcal{C}(c, f)](\mathbf{1}_c) = [Ff \circ \alpha_c](\mathbf{1}_c)$$

$\mathcal{C}(c, f)$ acts by pre-composition: $\mathcal{C}(c, f)(\mathbf{1}_c) = f$

$$\alpha_y(f) = (Ff)(\alpha_c(\mathbf{1}_c))$$

the right hand side does not depend on y thus <u>every</u> component α_y is determined by the value of component α_c evaluated at $\mathbf{1}_c$ thus the natural transformation is uniquely determined by

$$\alpha_c(\mathbf{1}_c) \in Fc$$

4. in the other direction every natural transformation $\alpha : \mathcal{C}(c, -) \Rightarrow F$ can be evaluated at $\mathbf{1}_c$ getting a point in Fc

$$\alpha_c(\mathbf{1}_c) \in Fc$$

□

The intuition is that the naturality condition is so strong that it is sufficient to know the value of α_c at $\mathbf{1}_c$ to <u>uniquely</u> determine the rest of the natural transformation.

Proof of naturality. Naturality of

$$[\mathcal{C}, \mathcal{S}et](\mathcal{C}(c, -), F) \cong Fc$$

in c and F means that they are the components ϕ of <u>two</u> natural isomorphisms with the following naturality squares[2]

1. keeping F fixed and taking a morphism $f : c \to c'$ it can be seen that the naturality square

[2] we use the notation $f_\star(g) = fg$ for post-composition and $f^\star(g) = gf$ for pre-composition

$$
\begin{array}{ccccc}
c & & [\mathcal{C},\mathcal{S}et](\mathcal{C}(c,-),F) & \xleftrightarrow{\phi_c} & Fc \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle (f^\star)^\star}\big\downarrow & & \big\downarrow{\scriptstyle Ff} \\
c' & & [\mathcal{C},\mathcal{S}et](\mathcal{C}(c',-),F) & \xleftrightarrow[\phi_{c'}]{} & Fc'
\end{array}
$$

commutes

2. keeping c fixed and taking a natural transformation $\eta : F \Rightarrow G$ it can be seen that the naturality square

$$
\begin{array}{ccccc}
F & & [\mathcal{C},\mathcal{S}et](\mathcal{C}(c,-),F) & \xleftrightarrow{\phi_F} & Fc \\
{\scriptstyle \eta}\big\downarrow & & {\scriptstyle \eta_\star}\big\downarrow & & \big\downarrow{\scriptstyle \eta_c} \\
G & & [\mathcal{C},\mathcal{S}et](\mathcal{C}(c,-),G) & \xleftrightarrow[\phi_G]{} & Gc
\end{array}
$$

commutes

□

See [1] Ch.2.2, [2] Ch.2.2, [3] Ch.15.

See also:

- Category Theory II 4.2: The Yoneda Lemma - YouTube
- The Yoneda Lemma

[1] E. Riehl, *Category Theory in Context* (Dover, 2015).
[2] P. Perrone, *Notes on Category Theory with Examples from Basic Mathematics*, (2019).
[3] B. Milewski, *Category Theory for Programmers* (2019).